Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Time 1 hour 30 minutes

Paper reference **9FM0/01**

Further Mathematics

Advanced

PAPER 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- Good luck with your examination.

Turn over

1. The transformation P is an enlargement, centre the origin, with scale factor k, where $k > 0$

The transformation Q is a rotation through angle θ degrees anticlockwise about the origin.

The transformation P followed by the transformation Q is represented by the matrix

$$\mathbf{M} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

(a) Determine

(i) the value of k,

(ii) the smallest value of θ

(4)

A square S has vertices at the points with coordinates $(0, 0)$, $(a, -a)$, $(2a, 0)$ and (a, a) where a is a constant.

The square S is transformed to the square S' by the transformation represented by $\mathbf{M}$.

(b) Determine, in terms of a, the area of S'

(2)

tranformation $P = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ transformation $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

↳ taken from formula for 'anticlockwise θ° rotation about the origin' in formula booklet

first of all recognise M as the single matrix formed from combining the two successive matrix transformations P and Q

$$M = QP$$

$$\Rightarrow M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

(i) METHOD 1: finding 'k' using square root of det(M)

know that for M representing an enlargement followed by a rotation, the only thing that will affect the ASF is the enlargement, hence using

$ASF = \det(M)$ (derived from: area of object $\times \det(M) =$ area of image)

$$ASF = \det\begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

using formula for det. of a 2x2 matrix

$$ASF = -4(-4) - (-4\sqrt{3} \times 4\sqrt{3})$$

$$= 16 - (-48)$$
$$= 16 + 48$$
$$= 64$$

but 'k' refers to the L.S.F of the enlargement

using $LSF = \sqrt{ASF}$
$$= \sqrt{64}$$
$$= 8$$
$$\boxed{\therefore k = 8}$$

(ii) following from METHOD 1, finding 'θ' using formula book equation

now we know that $k=8$ we can account for this 'enlargement'

by dividing M by 8 and getting Q

$$\frac{M}{8} = \begin{pmatrix} \frac{-4}{8} & \frac{-4\sqrt{3}}{8} \\ \frac{4\sqrt{3}}{8} & \frac{-4}{8} \end{pmatrix}$$

$$Q = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

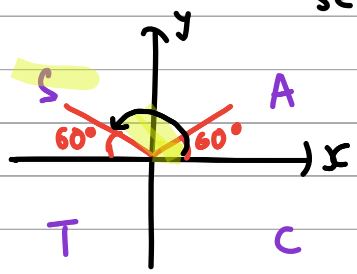
now to get the θ° of rotation, need to COMPARE CORRESPONDING ELEMENTS from the formula book equ. and the Q above

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$\cos\theta = -\frac{1}{2} < 0$

$\sin\theta = \frac{3}{2} > 0$

looking at CAST DIAGRAM to see where $\cos\theta < 0$ and $\sin\theta > 0$ = II quadrant



$\therefore$ first finding 'principal angle'

$\cos^{-1}(\frac{1}{2}) = 60°$

OR $\sin^{-1}(\frac{3}{2}) = 60°$

and finding the corresponding angle in the II quadrant

$180° - 60° = 120°$

$$\boxed{\therefore \theta = 120°}$$

METHOD 2: finding 'k' and 'θ' using matrix multiplication

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Multiply the elements in the row of the first matrix by the elements in the column of the second matrix and sum in between

$$= \begin{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \\ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta(k) - \sin\theta(k) & \cos\theta(0) - \sin\theta(0) \\ \sin\theta(k) + \cos\theta(0) & \sin\theta(0) + \cos\theta(k) \end{pmatrix}$$

$$= \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & 0 \end{pmatrix}$$

and equating to M in question

$$\begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix} = \begin{pmatrix} -4 & -4\sqrt{3} \\ 4\sqrt{3} & -4 \end{pmatrix}$$

...comparing corresponding elements:

$$k\cos\theta = -4 \quad -①$$

$$k\sin\theta = 4\sqrt{3} \quad -②$$

and solve simultaneously for 'θ' and 'k'

② ÷ ① (to eliminate 'k')

$$\frac{\cancel{k}\sin\theta}{\cancel{k}\cos\theta} = \frac{\cancel{4}\sqrt{3}}{-\cancel{4}}$$

$$\Rightarrow \tan\theta = -\sqrt{3}$$

Solve for θ

$$\theta = \tan^{-1}(-\sqrt{3})$$

$$= -60°$$

using angle laws (to get +ve angle)

$$= -60° + 180°$$

Question 1 continued

$\therefore \boxed{\theta = 120°}$

subbing θ into any of ① or ② to get 'k'

$k\sin(120°) = 4\sqrt{3}$

$\Rightarrow k\,\frac{\sqrt{3}}{2} = 4\sqrt{3}$

$\div \frac{\sqrt{3}}{2}$ $\quad \div \frac{\sqrt{3}}{2}$

$\boxed{\Rightarrow k = 8}$

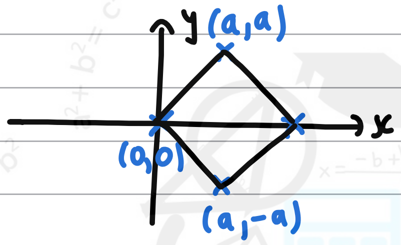
(b) know the formula area of object $\times$ det(M) = area of image
(S) (S')

and know that $\det(M) = (\text{"}k\text{"})^2$

$= (8)^2$

$= 64$

$\therefore$ need to find area of object; drawing:



WAY 1: using (magnitude of a side $\times 2$)

eg. $\sqrt{(a-0)^2+(a-0)^2}$

$= \sqrt{a^2+a^2} = \sqrt{2a^2}$

$\therefore \text{area} = \sqrt{2a^2} \times \sqrt{2a^2}$

$= 2a^2$

OR

WAY 2: splitting into 2 triangles side length 'a' and '2a'

$\cancel{2} \times \frac{1}{\cancel{2}}(a \times 2a)$

$= 2a^2$

$\therefore$ area of $S' = 2a^2 \times 64$

$\boxed{= 128a^2}$

(Total for Question 1 is 6 marks)

2. (a) Use the Maclaurin series expansion for $\cos x$ to determine the series expansion of $\cos^2\left(\frac{x}{3}\right)$ in ascending powers of x, up to and including the term in x^4

Give each term in simplest form.

(2)

(b) Use the answer to part (a) and calculus to find an approximation, to 5 decimal places, for

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx$$

(3)

(c) Use the integration function on your calculator to evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{x}\cos^2\left(\frac{x}{3}\right)\right) dx$$

Give your answer to 5 decimal places.

(1)

(d) Assuming that the calculator answer in part (c) is accurate to 5 decimal places, comment on the accuracy of the approximation found in part (b).

(1)

(a) METHOD 1: using Maclaurin expansion for $\cos\left(\frac{x}{3}\right)$ ALL SQUARED

recognising $\cos^2\left(\frac{x}{3}\right)$ is just the simple compound function $\cos\left(\frac{x}{3}\right)$ all SQUARED

$\therefore$ using $\cos x$ series expansion

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ (in FORMULA BOOKLET)

and using the substitution: $x \to x/3$

$\cos(x/3) = 1 - \frac{(x/3)^2}{2!} + \frac{(x/3)^4}{4!} - \dots$

expanding brackets and taking the denominator of the fractions in the numerator to the bottom

$\cos\left(\frac{x}{3}\right) = 1 - \frac{x^2}{2!(3^2)} + \frac{x^4}{4!(3^4)} - \dots$ "up to and including x^4"

Question 2 continued

$$= 1 - \frac{x^2}{18} + \frac{x^4}{1944} - \ldots$$

finally squaring to give $\cos^2\left(\frac{x}{3}\right)$

$$\cos^2\left(\frac{x}{3}\right) = \left(1 - \frac{x^2}{18} + \frac{x^4}{1944} - \ldots\right)^2$$

$$= \left(1 - \frac{x^2}{18} + \frac{x^4}{1944} - \ldots\right)\left(1 - \frac{x^2}{18} + \frac{x^4}{1944} - \ldots\right)$$

expanding the brackets

$$= 1 - \frac{x^2}{18} + \frac{x^4}{1944} - \frac{x^2}{18} + \frac{x^4}{324} + \frac{x^4}{1944}$$

collecting like terms

$$= 1 - \frac{\cancel{2}^{1}x^2}{\cancel{18}_{9}} + \frac{\cancel{2}^{1}x^4}{\cancel{1944}_{972}} + \frac{x^4}{324}$$

$$= 1 - \frac{x^2}{9} + \frac{x^4}{972} + \frac{3x^4}{972}$$

$$= 1 - \frac{x^2}{9} + \frac{\cancel{4}^{1}x^4}{\cancel{972}_{243}}$$

$$\boxed{= 1 - \frac{x^2}{9} + \frac{x^4}{243} - \ldots}$$

METHOD 2: using cos double angle identity and simple compound function $\cos\left(\frac{2x}{3}\right)$ Maclaurin's series

using cos double angle : $\cos 2x = 2\cos^2 x - 1$

and rearranging to make $\cos^2 x$ the subject

$$2\cos^2 x = 1 + \cos 2x \quad \div 2 \quad \div 2$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

question is asking for $\cos^2\left(\frac{x}{3}\right)$ series expansion, so using the substitution $x \to \frac{x}{3}$

$$\cos^2\left(\frac{x}{3}\right) = \frac{1}{2}\left(1 + \cos\left(\frac{2x}{3}\right)\right)$$

now replacing $\cos\left(\frac{2x}{3}\right)$ with the series expansion of $\cos x$ but where $x \to \frac{2x}{3}$

$$\frac{1}{2}\left(1 + \left(1 - \frac{1}{2!}\left(\frac{2x}{3}\right)^2 + \frac{1}{4!}\left(\frac{2x}{3}\right)^4 - \ldots\right)\right)$$

expanding inner brackets

$$\frac{1}{2}\left(1 + \left(1 - \frac{4x^2}{2!(3^2)} + \frac{16x^4}{4!(3^4)} - \ldots\right)\right)$$

Simplifying fractions

Question 2 continued

$$\Rightarrow \frac{1}{2}\left(1+\left(1-\frac{4x^2}{18}+\frac{16x^4}{1944}-\dots\right)\right)$$

multiplying out next brackets

$$\Rightarrow \frac{1}{2}\left(1+1-\frac{4x^2}{18}+\frac{16x^4}{1944}\right)$$

$$\Rightarrow \frac{1}{2}\left(2-\frac{\cancel{4}^{2}x^2}{\cancel{18}_{9}}+\frac{\cancel{16}^{2}x^4}{\cancel{1944}_{243}}\right)$$

$$\Rightarrow \frac{1}{2}\left(2-\frac{2x^2}{9}+\frac{2}{243}x^4\right)$$

multiplying by ½

$$\Rightarrow \boxed{1-\frac{x^2}{9}+\frac{1}{243}x^4\dots}$$

(b) subbing part (a)'s series expansion into the given integral

$$\int_{\pi/6}^{\pi/2}\left(\frac{1}{x}\left(1-\frac{x^2}{9}+\frac{x^4}{243}\right)\right)dx$$

expanding integral brackets

$$\int_{\pi/6}^{\pi/2}\left(\frac{1}{x}-\frac{x}{9}+\frac{x^3}{243}\right)dx$$

integrating the x terms; remembering the general result:

$$\int\frac{1}{x}\,dx=\ln|x|+c$$

and that $\int x^n=\frac{x^{n+1}}{n+1}+c$

$$=\left[\ln|x|-\frac{x^2}{18}+\frac{x^4}{243(4)}\right]_{\pi/6}^{\pi/2}$$

$$=\left[\ln|x|-\frac{x^2}{18}+\frac{x^4}{972}\right]_{\pi/6}^{\pi/2}$$

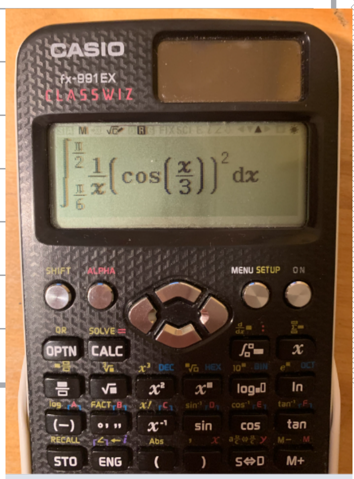
$$\Rightarrow\left\{\left[\ln(\pi/2)-\frac{(\pi/2)^2}{18}+\frac{(\pi/2)^4}{972}\right]-\left[\ln(\pi/6)-\frac{(\pi/6)^2}{18}+\frac{(\pi/6)^4}{972}\right]\right\}$$

evaluate this on calc

$=0.9829514\dots$

$\boxed{=0.98295\ (5\text{d.p})}$

(c) using CALC - "$\int_\square^\square\square$" button and 'x' button - in RADIANS



$=0.9828012\dots$

$\boxed{=0.98280\ (5\text{d.p})}$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 2 continued

(d) comparing part (b)'s 0.98295 with part (c)'s 0.98280,
see how part (b) only correct to 3 d.p

OR

$$\% \text{ error} = \frac{\text{part(b)} - \text{part(c)}}{\text{part(c)}} \times 100$$

$$= \frac{0.98295\ldots - 0.98280\ldots}{0.9828012\ldots} \times 100$$

$$\boxed{= 0.0153\% \text{ error}}$$

(Total for Question 2 is 7 marks)

3. The cubic equation

$$ax^3 + bx^2 - 19x - b = 0$$

where a and b are constants, has roots α, β and γ

The cubic equation

$$w^3 - 9w^2 - 97w + c = 0$$

where c is a constant, has roots $(4\alpha - 1)$, $(4\beta - 1)$ and $(4\gamma - 1)$

Without solving either cubic equation, determine the value of a, the value of b and the value of c.

(6)

noticing that the cubic equation given in terms of w is a LINEAR TRANSFORMATION of the cubic equation in terms of x using the substitution $w = 4x - 1$

∴ can utilise this to give us the values of 'a', 'b' and 'c'

METHOD 1: subbing $w = 4x + 1$ into the 'transformed' equation - although would naturally lean towards subbing into original equation - the 'transformed' has less UNKNOWNS ∴ this METHOD 1 would be quicker

$$(4x-1)^3 - 9(4x-1)^2 - 97(4x-1) + c = 0$$

...binomially expanding:

...Pascal's triangle:

1
1 2 1
1 3 3 1

$$(4x-1)^3 = (4x)^3 + 3((4x)^2(-1)) + 3((4x)(-1)^2) + (-1)^3]$$

$$= 64x^3 + 3(-16x^2) + 3(4x) - 1$$

$$= 64x^3 - 48x^2 + 12x - 1$$

$$(4x-1)^2 = (4x)^2 + 2((4x)(-1)) + (-1)^2$$

$$= 16x^2 - 8x + 1$$

$$-97(4x-1) = -388x + 97$$

subbing into above

$$64x^3 - 48x^2 + 12x - 1 - 9(16x^2 - 8x + 1) - 97(4x - 1) + c = 0$$

expanding brackets

$$64x^3 - 48x^2 + 12x - 1 - 144x^2 + 72x - 9 - 388x + 97 + c = 0$$

collect like terms

$$64x^3 - 192x^2 \boxed{-304x} + 87 + c = 0 \quad -①$$

P 6 6 7 9 6 A 0 1 0 3 6

Question 3 continued

which Should be the SCALED VERSION of :

$$ax^3+bx^2\boxed{-19x}-b=0 \quad -(2)$$

to find scale let's use the x terms:

$$\frac{-304}{-19}=16$$

∴ if divided (1) by 16 - we'll get (1) = (2)

(1) ÷ 16

$$\Rightarrow 4x^3-12x^2-19x+\frac{87+c}{16}=0 \quad -(3)$$

NOW comparing coefficients of (2) and (3)

... x^3:

$a=1$

... x^2:

$b=-12$

... constants:

$$\frac{87+c}{16}=-\text{"}-12\text{"}$$

$$\Rightarrow \frac{87+c}{16}=12$$

×16 ×16

$$\Rightarrow 87+c=192$$

$$\Rightarrow c=10$$

$$\boxed{\therefore a=4,\ b=-12,\ c=10}$$

METHOD 2: subbing $w=4x-1$ into original equation

$w=4x-1$

$\Rightarrow x=\frac{w+1}{4}$

$$a\left(\frac{w+1}{4}\right)^3+b\left(\frac{w+1}{4}\right)^2-19\left(\frac{w+1}{4}\right)-b=0$$

... taking denominator to the front and Binomially expanding the rest:

$$\frac{a}{4^3}(w+1)^3+\frac{b}{4^2}(w+1)^2-\frac{19}{4}(w+1)-b=0$$

... Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

$(w+1)^3=w^3+3w^2+3w+1$

$(w+1)^2=w^2+2w+1$

Question 3 continued $-19(w+1) = -19w-19$

$$\Rightarrow \frac{a}{64}(w^3+3w^2+3w+1) + \frac{b}{16}(w^2+2w+1) - \frac{19}{4}(w+1) - b = 0$$

expand brackets

$$\frac{a}{64}w^3 + \frac{3a}{64}w^2 + \frac{3a}{64}w + \frac{a}{64} + \frac{b}{16}w^2 + \frac{2b}{16}w + \frac{b}{16} - \frac{19}{4}w - \frac{19}{4} - b = 0$$

collect like terms

$$\frac{a}{64}w^3 + \left(\frac{3a}{64} + \frac{b}{16}\right)w^2 + \left(\frac{3a}{64} + \frac{2b}{16} - \frac{19}{4}\right)w + \left(\frac{a}{64} + \frac{b}{16} - \frac{19}{4} - b\right) = 0$$

getting common denominator

$$\frac{a}{64}w^3 + \left(\frac{3a+4b}{64}\right)w^2 + \left(\frac{3a+8b-304}{64}\right)w + \left(\frac{a+4b-304-64b}{64}\right) = 0$$

Simplify

$$\frac{a}{64}w^3 + \left(\frac{3a+4b}{64}\right)w^2 + \left(\frac{3a+8b-304}{64}\right)w + \left(\frac{a-60b-304}{64}\right) = 0$$

$\times \frac{64}{a}$ $\times \frac{64}{a}$

$$\Rightarrow w^3 + \left(\frac{3a+4b}{a}\right)w^2 + \left(\frac{3a+8b-304}{a}\right)w + \left(\frac{a-60b-304}{a}\right) = 0$$

now compare coefficients with transformed equation:

$$w^3 - 9w^2 - 97w + c = 0$$

$\therefore w^2$:

$$\frac{3a+4b}{a} = -9$$

$\times a$ $\times a$

$$3a + 4b = -9a$$

$$\Rightarrow 12a + 4b = 0 \quad -①$$

$\therefore w$:

$$\frac{3a+8b-304}{a} = -97$$

$\times a$ $\times a$

$$3a + 8b - 304 = -97a$$

$$\Rightarrow 100a + 8b = 304 \quad -②$$

solve simultaneously (equ. solver CALC or by elimination)

② $- 2\times$①

$$\begin{array}{r} 100a + 8b = 304 \\ -\quad 24a + 8b = 0 \\ \hline 76a = 304 \end{array}$$

$\div 76$ $\div 76$

$$\Rightarrow a = 4$$

sub into any of ① or ②

$$12(\text{"}4\text{"}) + 4b = 0$$

$$48 + 4b = 0$$

$$\Rightarrow 4b = -48$$

$$\Rightarrow b = -12$$

P 6 6 7 9 6 A 0 1 2 3 6

Question 3 continued

$\therefore$ integers:

$$\frac{a-60b-304}{a}=c$$

$$\frac{\text{"}4\text{"}-60(\text{"}-12\text{"})-304}{a}=c$$

$$\Rightarrow c=\frac{420}{4}=105$$

$$\boxed{\therefore\ a=4,\ b=-12,\ c=105}$$

METHOD 3: using the transformed roots of polynomials formulae

· first for original:

$$\underset{\sum\alpha}{a}x^3+\underset{\sum\alpha\beta}{b}x^2\underset{\alpha\beta\gamma}{-19}x-b=0$$

$$\sum\alpha=-b/a,\ \sum\alpha\beta=-19/a,\ \alpha\beta\gamma=b/a$$

· now for transformed:

$$w^3-\underset{\sum\alpha}{9}w^2-\underset{\sum\alpha\beta}{97}w+\underset{\alpha\beta\gamma}{c}=0\ \Rightarrow\ \sum\alpha=9,\ \sum\alpha\beta=-97,\ \alpha\beta\gamma=-c$$

① first evaluating 'sum of roots'

$$\sum\alpha=4\alpha-1+4\beta-1+4\gamma-1$$

$$9=4(\sum\alpha)-3$$

$$9=4(-b/a)-3$$

$$\Rightarrow -\frac{4b}{a}=12$$

$\times a \quad \times a$

$$-4b=12a$$

$\div -4 \quad \div -4$

$$\Rightarrow b=-3a \quad -①$$

② next evaluate sum of product pairs

$$\sum\alpha\beta=(4\alpha-1)(4\beta-1)+(4\alpha-1)(4\gamma-1)+(4\beta-1)(4\gamma-1)$$

$$=16\alpha\beta-4\alpha-4\beta+1+16\alpha\gamma-4\alpha-4\gamma+1+16\beta\gamma-4\beta-4\gamma+1$$

$$-97=16(\sum\alpha\beta)-8(\sum\alpha)+3$$

$$=16\left(\frac{-19}{a}\right)-8\left(\text{"}\frac{3\cancel{a}}{\cancel{a}}\text{"}\right)+3$$

$$\Rightarrow 16\left(-\frac{19}{a}\right)-8(3)+3=-97$$

$$\frac{-304}{a}=-97+24-3$$

$$\frac{-304}{a}=-76$$

$\times a \quad \times a$

$$-304=-76a$$

$\div -76 \quad \div -76$

$$a=4$$

$\therefore$ subbing into ①

(Total for Question 3 is 6 marks)

$$b = -3(4)$$
$$= -12$$

③ finally product of roots

$$\alpha\beta\gamma = (4\alpha - 1)(4\beta - 1)(4\gamma - 1)$$
$$-c = (16\alpha\beta - 4\alpha - 4\beta + 1)(4\gamma - 1)$$
$$= 64\alpha\beta\gamma - 16\alpha\gamma - 16\beta\gamma + 4\gamma - 16\alpha\beta + 4\alpha + 4\beta - 1$$
$$= 64(\alpha\beta\gamma) - 16(\Sigma\alpha\beta) + 4(\Sigma\alpha) - 1$$
$$= 64\left(\frac{``-12"}{``-4"}\right) - 16\left(\frac{-19}{``4"}\right) + 4\left(-\frac{``-12"}{4}\right) - 1$$
$$= -192 - 16\left(-\frac{19}{4}\right) + 12 - 1$$
$$\Rightarrow -c = -105$$
$$\Rightarrow c = 105$$

$$\boxed{\therefore a = 4,\ b = -12,\ c = 105}$$

4. (i) **A** is a 2 by 2 matrix and **B** is a 2 by 3 matrix.

Giving a reason for your answer, explain whether it is possible to evaluate

(a) **AB**

(b) **A** + **B**

(2)

(ii) Given that

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \lambda\mathbf{I}$$

where a, b and λ are constants,

(a) determine

- the value of λ
- the value of a
- the value of b

(b) Hence deduce the inverse of the matrix $\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}$

(3)

(iii) Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & \cos\theta \\ 0 & \cos 2\theta & \sin 2\theta \end{pmatrix} \qquad \text{where } 0 \leqslant \theta < \pi$$

determine the values of θ for which the matrix **M** is singular.

(4)

(i)

(a) it is possible (A and B are multiplicatively conformable) ∵ no. of columns of A = no. of rows of B

(b) it isn't possible (A and B are not additively conformable) ∵ A and B have different dimensions

(ii) evaluating first the IDENTITY MATRIX i.e a zero matrix with 'λ's in main diagonal

$$\begin{pmatrix} -5 & 3 & 1 \\ a & 0 & 0 \\ b & a & b \end{pmatrix}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

then use matrix multiplication - multiply the elements in the row of the first matrix by the elements in the column of the second matrix and sum in between - BUT BE SELECTIVE ON WHAT YOU MULTIPLY

Question 4 continued

· for the value of 'λ' see how should do row 1 multiplied by column 1 as this would have no unknowns on the L.H.S and just 'λ' on the R.H.S

$\Rightarrow -5(0)+3(2)+1(-1)=\lambda$

$\Rightarrow \boxed{\lambda=5}$

· for the value of 'a', try multiplying row 2 by column 2 as doesn't involve unknown 'b' AND makes sure 'a' isn't being multiplied by 0

$\Rightarrow a(5)+0(12)+0(-11)=\text{"}5\text{"}$

$\Rightarrow 5a=5$

$\div 5 \quad \div 5$

$\boxed{\Rightarrow a=1}$

· for 'b' can either:

...multiply row 3 by column 1:

$(b)(0)+(a)(2)+(b)(-1)=0$

$\Rightarrow 2a-b=0$

$2(\text{"}1\text{"})=b$

$\Rightarrow \boxed{b=2}$

...row 3 by column 2:

$(b)(5)+\text{"}1\text{"}(12)+(b)(-11)=0$

$\Rightarrow 12=6b$

$\div 6 \quad \div 6$

$\boxed{b=2}$

...row 3 by column 3:

$b(0)+\text{"}1\text{"}(-1)+b(3)=\text{'}5\text{'}$

$\Rightarrow 3b-1=5$

$3b=6$

$\div 3 \quad \div 3$

$\boxed{b=2}$

(b) METHOD 1: using $MM^{-1}=M^{-1}M=I$ (BEST METHOD under TIMED CONDITIONS)

let $A=\begin{pmatrix}-5 & 3 & 1\\ a & 0 & 0\\ b & a & b\end{pmatrix}$

subbing in 'a' and 'b' values from (a)

$A=\begin{pmatrix}-5 & 3 & 1\\ \text{"}1\text{"} & 0 & 0\\ \text{"}2\text{"} & \text{"}1\text{"} & \text{"}2\text{"}\end{pmatrix}$

now notice that to deduce M^{-1}, need the product of the two matrices in part (a) to be $AA^{-1}=I$

∴ if we just divide the other matrix by λ

$\begin{pmatrix}0 & 5 & 0\\ 2 & 12 & -1\\ -1 & -11 & 3\end{pmatrix}\div 5$

then this would mean

$A\times\frac{1}{5}\begin{pmatrix}0 & 5 & 0\\ 2 & 12 & -1\\ -1 & -11 & 3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$\Rightarrow$

$$\boxed{A^{-1}=\frac{1}{5}\begin{pmatrix}0 & 5 & 0\\ 2 & 12 & -1\\ -1 & -11 & 3\end{pmatrix}}$$

METHOD 2 : calc classwiz

4 . MATRIX - Define Mat A - OPTN MAT A - 'x^{-1}' button

and get $\boxed{A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0.4 & 2.4 & -0.2 \\ -0.2 & -2.2 & 0.6 \end{pmatrix}}$

METHOD 3 : manually inverse

NOTE : this is a method NOT recommended for just 1 mark but it IS awarded in the mark scheme

step 1 :

$\det(A) = -5\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$= -5(0) - 3(2) + 1(1)$

$= -5$

step 2 : matrix of minors (2×2 matrix left after deleting all rows and columns that the chosen element is in)

$M = \begin{pmatrix} \overset{+}{0} & \overset{-}{2} & \overset{+}{1} \\ \overset{-}{5} & \overset{+}{-12} & \overset{-}{-11} \\ \overset{+}{0} & \overset{-}{-1} & \overset{+}{-3} \end{pmatrix}$

step 3 : matrix of cofactors (changing signs of elements marked with '−')

$C = \begin{pmatrix} 0 & -2 & 1 \\ -5 & -12 & 11 \\ 0 & 1 & -3 \end{pmatrix}$

step 4 : transpose highlighted

$C^T = \begin{pmatrix} 0 & -5 & 0 \\ -2 & -12 & 1 \\ 1 & 11 & -3 \end{pmatrix}$

step 5 : inverse $= \frac{1}{\det(A)} C^T$

$= \frac{1}{-5}\begin{pmatrix} 0 & -5 & 0 \\ -2 & -12 & 1 \\ 1 & 11 & -3 \end{pmatrix}$

$\times -1 \qquad \times -1$

$= \frac{1}{5}\begin{pmatrix} 0 & 5 & 0 \\ 2 & 12 & -1 \\ -1 & -11 & 3 \end{pmatrix}$

(iii) remembering that for matrix to be singular, $\det(A) = 0$

∴ using formula for det. of 3×3 matrices

$\det(A) = 0$

$$0 = 1\begin{vmatrix} \sin\theta & \cos\theta \\ \cos 2\theta & \sin 2\theta \end{vmatrix} - 1\begin{vmatrix} 0 & \cos\theta \\ 0 & \sin 2\theta \end{vmatrix} + 1\begin{vmatrix} 0 & \sin\theta \\ 0 & \cos 2\theta \end{vmatrix}$$

$$0 = 1(\sin\theta\sin 2\theta - \cos\theta\cos 2\theta) - (0) + 1(0)$$

$\Rightarrow 0 = \sin\theta\sin 2\theta - \cos\theta\cos 2\theta$

solve to find 'θ'

METHOD 1: cos compound angle formula

know that $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$0 = -(\cos(2\theta + \theta))$

$0 = -\cos(3\theta)$

$\Rightarrow \cos(3\theta) = 0$

recognise compound angle $\therefore$ need to solve

manipulate range

let $y = 3\theta$

$0 \leq y \leq 3\pi$

$\cos y = 0$

$y = \cos^{-1}(0)$

$= \pi/2, (2\pi - \pi/2 =) \ 3\pi/2$

$+2\pi \qquad +2\pi$

$= \frac{5}{2}\pi, \cancel{\frac{7\pi}{2}}$

$\therefore y = \pi/2, 3\pi/2, \frac{5\pi}{2}$

but $y = 3\theta$

$\Rightarrow \theta = \frac{y}{3}$

$$\boxed{\therefore \theta = \pi/6, \pi/2, 5\pi/6}$$

METHOD 2: using cos double angle formula and Pythagorean identity

using $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$0 = \sin(2\sin\theta\cos\theta) - \cos\theta(2\cos^2\theta - 1)$

$\Rightarrow 0 = 2\sin^2\theta\cos\theta - 2\cos^3\theta + \cos\theta$

using Pythagorean identity

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

$0 = 2(1 - \cos^2\theta)\cos\theta - 2\cos^3\theta + \cos\theta$

expand brackets

$= 2\cos\theta - 2\cos^3\theta - 2\cos^3\theta + \cos\theta$

collect like terms

$\Rightarrow 3\cos\theta - 4\cos^3\theta = 0$

DO NOT WRITE IN THIS AREA

Question 4 continued

FACTORISE $\cos\theta$ out

$\cos\theta(3-4\cos^2\theta)=0$

and make each bracket equal to 0

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2,\ (2\pi - \pi/2 =)\ \cancel{3\pi/2}$

Out of range

$3-4\cos^2\theta=0$

$\Rightarrow 4\cos^2\theta = 3 \quad \div 4$

$\div 4 \quad \cos^2\theta = \frac{3}{4}$

square root both sides

$\Rightarrow \cos\theta = \pm\sqrt{3/4}$

...+ve:

$\cos\theta = \sqrt{3/4}$

$\theta = \cos^{-1}(\sqrt{3/4})$

$= \pi/6,\ (2\pi - \pi/6 =)\ \cancel{11\pi/6}$

Out of range

...-ve:

$\cos\theta = -\sqrt{3/4}$

$\theta = \cos^{-1}(-\sqrt{3/4})$

$= 5\pi/6$

$\boxed{\therefore \theta = \pi/6,\ \pi/2,\ 5\pi/6}$

(Total for Question 4 is 9 marks)

5. (i) Evaluate the improper integral

$$\int_1^{\infty} 2\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x$$

(3)

(ii) The air temperature, θ °C, on a particular day in London is modelled by the equation

$$\theta = 8 - 5\sin\left(\frac{\pi}{12}t\right) - \cos\left(\frac{\pi}{6}t\right) \qquad 0 \leqslant t \leqslant 24$$

where t is the number of hours after midnight.

(a) Use calculus to show that the mean air temperature on this day is 8 °C, according to the model.

(3)

Given that the actual mean air temperature recorded on this day was higher than 8 °C,

(b) explain how the model could be refined.

(1)

(i) notice how integral is 'improper' due to the '∞' in the limits, but if we can get the integral to converge to a finite area then we can still evaluate it and get a real answer

first integrating indefinitely:

$$\int 2e^{-\frac{1}{2}x}\,dx$$

using 'general result' for exponentials i.e divide by derivative of power and keep power as it is

$$\left[\frac{2}{-\frac{1}{2}}e^{-\frac{1}{2}x}\right] + c$$

$$= \left[-4e^{-\frac{1}{2}x}\right] + c$$

now applying the limits

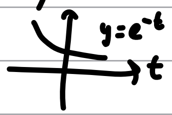
$$\int_1^{\infty} 2e^{-\frac{1}{2}x}\,dx$$

$$= \lim_{t\to\infty}\left[-4e^{-\frac{1}{2}x}\right]_1^t$$

$$= \lim_{t\to\infty}\left\{\left(-4e^{-\frac{1}{2}t}\right) - \left(-4e^{-\frac{1}{2}(1)}\right)\right\}$$

$$= \lim_{t\to\infty}\left(-4e^{-\frac{1}{2}t} + 4e^{-\frac{1}{2}}\right)$$

as $t\to\infty$, $e^{-\frac{1}{2}t} \to 0$



$\Rightarrow -4e^{-\frac{1}{2}t} \to 0$

P 6 6 7 9 6 A 0 1 8 3 6

Question 5 continued

$\therefore$ integral converges to $\boxed{4e^{-1/2}}$

(ii)(a) remembering formula for mean of a function (and applying it to given model for air temperature)

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

subbing in $0 \le t \le 24$ as the limits and $f(x) = \theta$ given in the question

$$\frac{1}{24-0}\int_0^{24}\left(8-5\sin\left(\frac{\pi}{12}t\right)-\cos\left(\frac{\pi}{6}t\right)\right)dt$$

$$\Rightarrow \frac{1}{24}\int_0^{24}\left(8-5\sin\left(\frac{\pi}{12}t\right)-\cos\left(\frac{\pi}{6}t\right)\right)dt$$

using $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

and $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

$$\Rightarrow \frac{1}{24}\left[8t+\frac{60}{\pi}\cos\left(\frac{\pi}{12}t\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]_0^{24}$$

$$\Rightarrow \frac{1}{24}\left\{\left[8(24)+\frac{60}{\pi}\cos\left(\frac{\pi}{12}\times 24\right)-\frac{6}{\pi}\sin\left(\frac{\pi}{6}\times 24\right)\right]-\left[8(0)+\frac{60}{\pi}\cos\left(\frac{\pi}{12}\times 0\right)+\frac{6}{\pi}\sin\left(\frac{\pi}{6}(0)\right)\right]\right\}$$

$$\Rightarrow \frac{1}{24}\left(192+\cancel{\frac{60}{\pi}}-\frac{6}{\pi}(0)-\cancel{\frac{60}{\pi}}\right)$$

$$\Rightarrow \frac{1}{24}(192)$$

$$= \boxed{8\,°C}$$

(ii)(b) $\boxed{\text{eg. increase value of constant 8 / adapt constant 8 to function which takes values} > 8\,°C}$

(Total for Question 5 is 7 marks)

6. A tourist decides to do a bungee jump from a bridge over a river.
One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.
The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is $0\,\mathrm{m\,s^{-1}}$
- the acceleration of the tourist is $13.6\,\mathrm{m\,s^{-2}}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$$5k\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + 17x = 0 \qquad t \geqslant 0$$

where k is a positive constant.

(a) Determine the value of k **(2)**

(b) Determine the particular solution to the differential equation. **(7)**

(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point. **(2)**

(d) Give a limitation of the model. **(1)**

(a) notice how we have an incomplete, HOMOGENOUS 2 O.D.E - however substituting in the INITIAL CONDITIONS from equation can help us find the value of 'k'

using fact that x = displacement = -20

$\frac{dx}{dt}$ = velocity = 0

$\frac{d^2x}{dt^2}$ = acceleration = 13.6

subbing these into the equation

$5(13.6) + 2k(0) + 7(-20) = 0$

expand the brackets

$\Rightarrow 68k + 0 - 304 = 0$

$\Rightarrow 68k = 340$ $\div 68$ $\div 68$

$\Rightarrow k = 5$

P 6 6 7 9 6 A 0 2 0 3 6

Question 6 continued

(b) subbing in the $k=5$ into the differential equation

$$5(5)\frac{d^2x}{dt^2}+2(5)\frac{dx}{dt}+17x=0$$

$$\Rightarrow 25\frac{d^2x}{dt^2}+10\frac{dx}{dt}+17x=0$$

we are asked to SOLVE this homogenous 2 O.D.E

A.E : $25m^2+10m+17=0$

using equation solver on classwiz calc OR quadratic formula

$$m=\frac{-10\pm\sqrt{(10)^2-4(25)(17)}}{2(25)}$$

$$=\frac{-10\pm\sqrt{100-1700}}{50}$$

$$=\frac{-10\pm\sqrt{-1600}}{50}$$

$$=\frac{-10\pm 40i}{50}$$

$$=\frac{-1\pm 4i}{5} \Rightarrow \text{C.F}=-0.2\pm 0.8i$$

now using the complex conjugate result for 2 O.D.E (where $b^2-4ac<0$)

$\Rightarrow m=a\pm bi \therefore y=e^{ax}(A\cos bx+B\sin bx)$

G.S : $x=e^{-0.2t}(A\cos 0.8t+B\sin 0.8t)$

but this G.S represents a whole family of differential equations that have different 'A' and 'B' values - question wants a particular solution so subbing in the following initial conditions:

when $t=0$, $x=-20$

$$-20=e^{-0.2(0)}(A\cos(0.8\times 0)+B\sin(0.8\times 0))$$

$$\Rightarrow -20=1(A\cos(0)+B\sin(0))$$

$$\Rightarrow -20=A$$

when $t=0$, $\frac{dx}{dt}=0$

differentiating general solution

using diff. rules for exponentials: $\frac{d}{dx}e^{kx}=ke^{kx}$

Question 6 continued

and trig $\frac{d}{dx}\sin kx = k\cos kx$

$\frac{d}{dx}\cos kx = -k\sin kx$

(product rule)

$$\frac{dx}{dt} = e^{-0.2t}(-0.8A\sin 0.8t + 0.8B\cos 0.8t) + (-0.2e^{-0.2t})(A\cos 0.8t + B\sin 0.8t)$$

$$0 = e^{-0.2(0)}(-0.8A\sin(0) + 0.8B\cos(0)) + (-0.2e^{-0.2(0)})(A\cos(0) + B\sin(0))$$

$\Rightarrow$ $0 = 0.8B - 0.2A$

Sub in "$A = -20$"

$0 = 0.8B - 0.2(-20)$

$\Rightarrow 0 = 0.8B + 4$

$\Rightarrow 0.8B = -4$

$\div 0.8 \quad \div 0.8$

$\Rightarrow B = -5$

$\boxed{\therefore \text{particular solution is } y = e^{-0.2t}(-20\cos(0.8t) - 5\sin(0.8t))}$

(c) we can see x in question refers to the vertical displacement ABOVE a fixed point 30m above ground $\therefore$ to get FULL VERTICAL HEIGHT above river 15 secs after reached lowest point

need $30 + [e^{-0.2(15)}(-20\cos(0.8 \times 15) - 5\sin(0.8 \times 15))]$

expand brackets

$\Rightarrow 30 + [e^{-3}(-20\cos(12) - 5\sin(12))]$

evaluate on calc - in RADIANS

$\Rightarrow 29.293316...$

$\Rightarrow \boxed{29.3\text{m } (3\text{ s.f})}$

(d) possible limitations:

- unlikely that rope will remain taut
- model predicts that tourist will continue to move up and down (in fact, will eventually lose momentum)
- tourist modelled as a particle is UNREALISTIC

Question 6 continued

(Total for Question 6 is 12 marks)

7. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Show that vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is perpendicular to Π.

(2)

(b) Hence find a Cartesian equation of Π.

(2)

The line l has equation

$$\mathbf{r} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}$$

where t is a scalar parameter.

The point A lies on l.

Given that the shortest distance between A and Π is $2\sqrt{29}$

(c) determine the possible coordinates of A.

(4)

(a) remembering the fact that for a 3D VECTOR to be perpendicular to the plane π (given in vector parametric form i.e where $\pi = a + \lambda b + \lambda c$)

position vector — two non-parallel direction vectors

then the dot product of it and the two non-parallel vectors 'b' and 'c' must be 0 (comes from angle between 2 lines = 90° ∴ $\cos(90)° = \frac{a.b}{|a||b|} \Rightarrow a.b = 0$)

- remembering dot product as sum of product of each element

$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 2(-1) + 3(2) + (-4)(1) = 6 - 2 - 4 = 0$$

$$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2(2) + 0(3) - 4(1) = 4 - 4 = 0$$

$\boxed{\text{as } \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \text{ is perpendicular to both direction vectors of } \pi \text{, must be perpendicular to } \pi}$

(b) We know that the general Cartesian equation for a plane is:

$xn_1 + yn_2 + zn_3 = p$ where $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ are normal (perp.) to the plane

and 'p' comes from $\underset{\uparrow}{r} \cdot n = p$

position vector

· we already know the normal from part (a) $-\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}-$ now doing the position vector 'a' to get p

$$\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 3(2) + 3(3) + 2(-4) = 7$$

∴ completing the general Cartesian form, get:

$$\boxed{\Pi = 2x + 3y - 4z = 7}$$

(c) METHOD 1: using formula book equation

question is asking us to use and sub into the equation for shortest distance between a point and a plane :

the perp. distance of (α, β, γ) from $n_1x + n_2y + n_3z - d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2}}$

(GIVEN IN FORMULA BOOK)

where general point $A = \begin{pmatrix} 4+t \\ -5+6t \\ 2-3t \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$$2\sqrt{29} = \frac{|2(4+t) + 3(-5+6t) - 4(2-3t) - 7|}{\sqrt{(2)^2 + (3)^2 + (-4)^2}}$$

expand brackets

$$2\sqrt{29} = \frac{|8 + 2t - 15 - 18t - 8 + 12t - 7|}{\sqrt{29}}$$

$\times \sqrt{29}$ $\times \sqrt{29}$

Question 7 continued

$\Rightarrow 2(29) = |8+2t-15+18t-8+12t-7|$

collect like terms

$58 = |32t-22|$

evaluating the modulus

$58 = \pm(32t-22)$

... +ve:

$58 = 32t-22$

$\Rightarrow 32t = 80$ $\div 32$

$\div 32$ $\Rightarrow t = \frac{80}{32}$

$\Rightarrow t = 5/2$

... -ve:

$58 = -(32t-22)$

$\Rightarrow 58 = -32t+22$

$\Rightarrow 32t = 22-58$

$\Rightarrow 32t = -36$ $\div 32$

$\div 32$ $\Rightarrow t = -9/8$

now subbing in both in turn into the general coordinates for A:

$$A = \begin{pmatrix} 4+(2.5) \\ -5+6(2.5) \\ 2-3(2.5) \end{pmatrix} = \begin{pmatrix} 13/2 \\ 10 \\ -11/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 4+(-9/8) \\ -5+6(-9/8) \\ 2-3(-9/8) \end{pmatrix} = \begin{pmatrix} 23/8 \\ -47/4 \\ 43/8 \end{pmatrix}$$

$\therefore$ possible A coordinates:

$$\boxed{\left(\frac{13}{2}, 10, -\frac{11}{2}\right) \text{ or } \left(\frac{23}{8}, -\frac{47}{4}, \frac{43}{8}\right)}$$

METHOD 2: separating 'shortest distances'

first need 'shortest distance from origin to plane' i.e $\frac{p}{|n|}$ from $r \cdot n = p$

$|n| = \sqrt{(2)^2+(3)^2+(-4)^2}$

$= \sqrt{29}$

$\therefore \frac{7}{\sqrt{29}}$

next perpendicular distance from plane containing p.o.i to the origin

i.e $\frac{a \cdot n}{|n|}$

$= \begin{pmatrix} 4+t \\ -5+6t \\ 2-3t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

P 6 6 7 9 6 A 0 2 6 3 6

Question 7 continued

$= 2(4+t) + 3(-5+6t) - 4(2-3t)$

expand brackets

$= \cancel{8} + 2t - 15 + 18t - \cancel{8} + 12t$

$= 32t - 15$

$\therefore \frac{32t-5}{\sqrt{29}}$

$\therefore$ min. distance =

$$\left| \frac{a \cdot n}{|n|} - \frac{p}{|n|} \right| = \left| \frac{32t-15}{\sqrt{29}} - \frac{7}{\sqrt{29}} \right|$$

$$2\sqrt{29} = \left| \frac{32t-22}{\sqrt{29}} \right|$$

$\times\sqrt{29}$ $\times\sqrt{29}$

$2(29) = |32t-22|$

$\Rightarrow 58 = |32t-22|$

evaluate modulus

$\Rightarrow 58 = \pm(32t-22)$

... +ve:

$58 = 32t - 22$

$\Rightarrow 32t = 80$

$\div 32$ $\div 32$

$\Rightarrow t = 5/2$

... -ve:

$58 = -(32t-22)$

$58 = -32t + 22$

$\Rightarrow 32t = -58 + 22$

$\Rightarrow 32t = -36$ $\div 32$

$\div 32$ $\Rightarrow t = -\frac{36}{32}$

$\Rightarrow t = -9/8$

now subbing both 't' values into the general coordinate for A

$$A = \begin{pmatrix} 4 + (5/2) \\ -5 + 6(5/2) \\ 2 - 3(5/2) \end{pmatrix} = \begin{pmatrix} 13/2 \\ 10 \\ -11/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 + (-9/8) \\ -5 + 6(-9/8) \\ 2 - 3(-9/8) \end{pmatrix} = \begin{pmatrix} 23/8 \\ 10 \\ -11/2 \end{pmatrix}$$

$\therefore$ possible coordinates of A are

$$\boxed{\left(\frac{13}{2}, 10, -\frac{11}{2}\right) \text{ or } \left(\frac{23}{8}, -\frac{47}{4}, \frac{43}{8}\right)}$$

(Total for Question 7 is 8 marks)

8. Two different colours of paint are being mixed together in a container.

The paint is stirred continuously so that each colour is instantly dispersed evenly throughout the container.

Initially the container holds a mixture of 10 litres of red paint and 20 litres of blue paint.

The colour of the paint mixture is now altered by

- adding red paint to the container at a rate of 2 litres per second
- adding blue paint to the container at a rate of 1 litre per second
- pumping fully mixed paint from the container at a rate of 3 litres per second.

Let r litres be the amount of red paint in the container at time t seconds after the colour of the paint mixture starts to be altered.

(a) Show that the amount of red paint in the container can be modelled by the differential equation

$$\frac{dr}{dt} = 2 - \frac{r}{\alpha}$$

where α is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the mixture of paint in the container to consist of equal amounts of red paint and blue paint, according to the model. Give your answer to the nearest second.

(6)

It actually takes 9 seconds for the mixture of paint in the container to consist of equal amounts of red paint and blue paint.

(c) Use this information to evaluate the model, giving a reason for your answer.

(1)

(a) recognise this as a 'modelling with 1 O.D.E' question (hinted by the 'instantly dispersed throughout' in the question) ∴ following the format:-

· volume of paint = 20L + 10L = 30L

· concentration of red paint = $\frac{r}{30}$ = $\frac{r}{30}$

· rate of red in = 2L/sec

· rate of red out = $\cancel{3}^{1}$L/sec × $\frac{r}{\cancel{30}_{10}}$

$= \frac{r}{10}$ L/sec

∴ $\frac{dr}{dt}$ = rate in − rate out

$= 2 - \frac{r}{10}$ L/sec $\boxed{\therefore \alpha = 10}$

P 6 6 7 9 6 A 0 2 8 3 6

(b) now asked to solve the 1 O.D.E

WAY 1: using integrating factor

...rewriting in form:

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dr}{dt} + \frac{r}{10} = 2$$

· Straight away can see cannot solve using separation of variables as involves an addition rather than the product of 2 variables and $\frac{dy}{dx}$

· also see cannot use reverse product rule as

$$\frac{d}{dt}(r) = \frac{dr}{dt} \ (\checkmark)$$

$$\frac{dr}{dt} + \frac{r}{10} = 2$$

$$\frac{d}{dx}\left(\frac{1}{10}\right) \neq 1(x)$$

$\therefore$ not reverse product rule

hence only way - introduce integration factor: $e^{\int P\,dx}$

$$\Rightarrow e^{\int \frac{1}{10} dt} = e^{t/10}$$

multiplying through by $e^{t/10}$

$$\frac{d}{dt}(r) = \frac{dr}{dt} \ (\checkmark)$$

$$\frac{dr}{dt} e^{t/10} + \frac{r}{10} e^{t/10} = 2e^{t/10}$$

$$\frac{d}{dt}\left(e^{t/10}\right) = \frac{1}{10} e^{t/10}$$

NOW we can check for reverse product rule (above)

rewrite

$$\frac{d}{dt}\left(re^{t/10}\right) = 2e^{t/10}$$

integrate both sides and solve for 'r'

$$\int \frac{d}{dt}\left(re^{t/10}\right) dt = \int 2e^{t/10}\, dt$$

$\Rightarrow$ G.S: $re^{t/10} = 20e^{t/10} + c$

but asked for PARTICULAR SOLUTION

using at $t=0$, $r=10$

$$(10)e^{0/10} = 20e^{0/10} + c$$

$$\Rightarrow 10 = 20 + c$$

$$\Rightarrow c = -10$$

P.S: $re^{t/10} = 20e^{t/10} - 10$

Question 8 continued

next, for time taken for mixture to contain equal amounts of 'r' and 'b', need to use fact that total V = 30L and ∴ sub in $r=15$ into above P.S

$$(15)e^{t/10} = 20e^{t/10} - 10$$

collecting exponentials and solving for 't'

$$20e^{t/10} - 15e^{t/10} = 10$$

$$\Rightarrow 5e^{t/10} = 10$$

$\div 5$ $\div 5$

$$\Rightarrow e^{t/10} = 2$$

taking logs of both sides

$$\frac{t}{10} = \ln 2$$

$$\Rightarrow t = 10\ln 2$$

$$= 6.93147.$$

$$= \boxed{7 \text{ secs } (1 \text{ s.f})}$$

METHOD 2: separation of variables (easier! if spotted)

making the equation in part (a) to have one common denominator

$$\frac{dr}{dt} = \frac{20-r}{10}$$

and noticing how no longer an addition ∴ can separate the variables and integrate both sides

$\div(20-r)$ $\div(20-r)$

$$\int \frac{1}{20-r}\,dr = \int \frac{1}{10}\,dt$$

using $\int \frac{1}{x}\,dx = \ln|x| + c$ and $\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$

$$G.S = -\ln|20-r| = \frac{1}{10}t + c$$

now difficult to rearrange for 'r' ∴ sub in initial conditions to get P.S

at $t=0$, $r=10$

$$-\ln|20-10| = \frac{1}{10}(0) + c$$

$$\Rightarrow c = -\ln|10|$$

$$P.S = -\ln|20-r| = \frac{1}{10}t - \ln|10|$$

and for mixture to have equal 'r' and 'b') – sub in $r = \frac{30}{2} = 15$ into P.S

$$-\ln|20-15| = \frac{1}{10}t - \ln|10|$$

$$-\ln|5| = \frac{1}{10}t - \ln|10|$$

$$\ln|10| - \ln|5| = \frac{1}{10}t$$

DO NOT WRITE IN THIS AREA

Question 8 continued

using quotient log law

$\ln\left|\frac{10}{5}\right| = \frac{1}{10}t$

$\Rightarrow \ln|2| = \frac{1}{10}t$

solve for t

$\Rightarrow t = 10\ln(2)$

$= 6.93147.$

$= \boxed{7 \text{ secs}}$

(c) working out % error in light of the measured 9

$\frac{7-9}{9} \times 100$

$= -20\%$

$\boxed{\Rightarrow \text{model is 20\% off the actual value} \therefore \text{not a great model}}$

(Total for Question 8 is 9 marks)

9. (a) Use a hyperbolic substitution and calculus to show that

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh} x\right] + k$$

where k is an arbitrary constant.

(6)

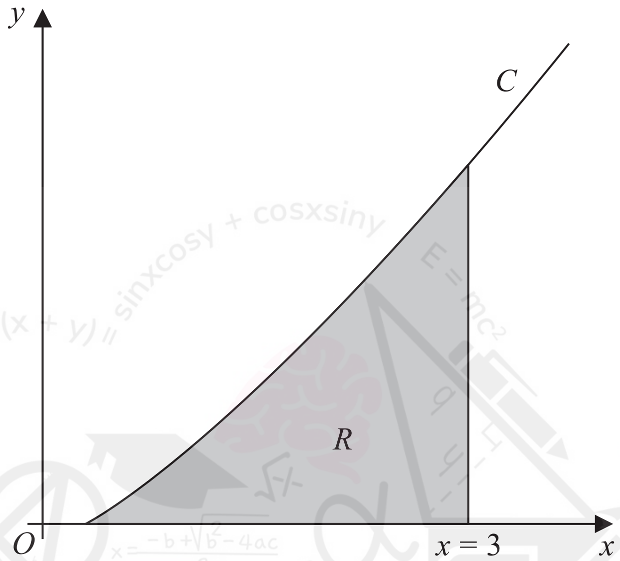


Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{4}{15}x \operatorname{arcosh} x \qquad x \geqslant 1$$

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Using algebraic integration and the result from part (a), show that the area of R is given by

$$\frac{1}{15}\left[17\ln\left(3+2\sqrt{2}\right) - 6\sqrt{2}\right]$$

(5)

(a) we're asked to show that

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh}(x)\right] + k$$

using the hyperbolic substitution : $x = \cosh u$

$\Rightarrow \frac{dx}{du} = \sinh u$

$\Rightarrow dx = \sinh u\, du$

and trying to replace functions in the integral with functions in terms of 'u'

$x^2 = \cosh^2 u$

and $\sqrt{x^2-1} = \sqrt{\cosh^2 u - 1}$

$\therefore$ using $\cosh^2 u - \sinh^2 u \equiv 1$

$\Rightarrow \cosh^2 u - 1 \equiv \sinh^2 u$

$= \sqrt{\sinh^2 u}$

$= \sinh u$

subbing into the integral

$\int \frac{\cosh^2 u}{\cancel{\sinh u}} \cancel{\sinh u}\, du$

$\int \cosh^2 u \, du$

but can't integrate trig powers $\therefore$ using TRIG IDENTITY

$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

$\Rightarrow \cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$

$= \frac{1}{2}(\cosh 2x + 1)$ $\therefore \cosh^2 u = \frac{1}{2}(\cosh 2u + 1)$

take the $\frac{1}{2}$ out

$= \frac{1}{2}\int \cosh 2u + 1 \, du$

using $\int \cosh x = \sinh x + c$

$= \frac{1}{2}\left[\frac{1}{2}\sinh 2u + u\right] + k$

noticing '½' and 'k' are correct as illustrated in 'show that',

but $\frac{1}{2}\sinh 2u$ and u need to be expressed

IN TERMS of 'x'

...for $\frac{1}{2}\sinh 2u$:

use hyperbolic version of sin double angle:

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$\therefore \frac{1}{2}\sinh 2u = \frac{1}{2}(2\sinh u \cosh u)$

$= \sinh u \cosh u$

know that $x = \cosh u$

and $\sinh u = \sqrt{\cosh^2 u - 1}$

$= \sqrt{x^2-1}$ (seen prev.)

$\therefore = x\sqrt{x^2-1}$

...for 'u':

use $x = \cosh u$

$\therefore u = \operatorname{arcosh} x$

$$\therefore \int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left[x\sqrt{x^2-1} + \operatorname{arcosh}x\right] + k$$

(b) for area bounded by C, need to evaluate:

$$\frac{4}{15}\int_0^3 x \operatorname{arcosh} x \, dx$$

noticing product of two functions ∴ hints at using integration by parts, using the following rule:

let u =
Logs
Inverse trig (FM)
Algebraic functions
Trig
Exponentials

$\therefore\ u = \operatorname{arcosh}x$ $\qquad v' = x$

$u' = \frac{1}{\sqrt{x^2-1}}$ (IN FORMULA BOOKLET) $\qquad v = \frac{x^2}{2}$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\Rightarrow \frac{4}{15}\left[\frac{1}{2}x^2\operatorname{arcosh}x - \int \frac{x^2}{2\sqrt{x^2-1}}\,dx\right]_0^3$$

$$\Rightarrow \frac{4}{15}\left[\frac{1}{2}x^2\operatorname{arcosh}x - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2-1}}\,dx\right]_0^3$$

↳ part (a)

$$\Rightarrow \frac{4}{15}\left[\frac{1}{2}x^2\operatorname{arcosh}x - \frac{1}{2}\left[\frac{1}{2}\left(x\sqrt{x^2-1} + \operatorname{arcosh}x\right)\right]\right]_0^3$$

expand square brackets

$$\Rightarrow \frac{4}{15}\left[\frac{1}{2}x^2\operatorname{arcosh}x - \frac{1}{4}x\sqrt{x^2-1} - \frac{1}{4}\operatorname{arcosh}x\right]_0^3$$

$$\Rightarrow \frac{4}{15}\left\{\left[\frac{1}{2}(3)^2\operatorname{arcosh}(3) - \frac{1}{4}(3)\sqrt{(3)^2-1} - \frac{1}{4}\operatorname{arcosh}(3)\right] - \left[(0)-(0)-(0)\right]\right\}$$

$$\Rightarrow \frac{4}{15}\left(\frac{9}{2}\operatorname{arcosh}(3) - \frac{3\sqrt{8}}{4} - \frac{1}{4}\operatorname{arcosh}(3)\right)$$

using DFN of arcoshx : $\ln\left|x + \sqrt{x^2+1}\right|$

$$\operatorname{arcosh}(3) = \ln\left|3 + \sqrt{(3)^2+1}\right|$$
$$= \ln\left|3 + \sqrt{8}\right|$$

$$\Rightarrow \frac{4}{15}\left(\frac{9}{2}\left(\ln(3+\sqrt{8})\right) - \frac{3\sqrt{8}}{4} - \frac{1}{4}\ln(3+\sqrt{8})\right)$$

Question 9 continued

taking 4 out

$$\frac{\cancel{4}}{15}\left(\frac{1}{\cancel{4}}\right)\left(18\ln(3+\sqrt{8}) - 3\sqrt{8} - \ln(3+\sqrt{8})\right)$$

$\sqrt{8} = 2\sqrt{2}$ $\quad 3\sqrt{8} = 3(2\sqrt{2})$

$= 6\sqrt{2}$

and collect like ln terms

$$\boxed{\Rightarrow \frac{1}{15}\left(17\ln(3+2\sqrt{2}) - 6\sqrt{2}\right)}$$

Question 9 continued

(Total for Question 9 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS